Turbulent Axial Dispersion Coefficients in the Liquid in a Two-Phase Bubble Column

I. H. LEHRER

Department of Chemical Engineering
Monash University
Clayton, Victoria 3168, Australia

INTRODUCTION

Axial dispersion in bubble columns has been discussed frequently in recent literature (e.g., Joshi, 1980; Field and Davidson, 1980; Riquartz, 1981; Baird and Rice and Joshi, 1981; reviews by Meersmann, 1977; Shah et al., 1978; Joshi and Shah, 1981).

The bubble columns discussed here are cylindrical sections filled with two-phase fluid in which liquid is the continuous phase and gas the dispersed phase. The columns are simple; i.e., the only significant bound to fluid motion is the column wall. There is steady upward mean flow of gas. Mean flow of liquid may be cocurrent with gas flow, countercurrent to gas flow, or the liquid may be in batch operation. Thus the mean flow of passing fluid is axial.

Considering a bubble column with turbulent liquid motion, the spread of residence times of passing fluid results mainly from eddies that produce convective axial mixing and from velocity profiles. For evaluation of the effects of nonuniform residence times, it is required to know the magnitude of the turbulent axial dispersion coefficient; for the liquid phase, this is E_L.

E_L can be defined by the following equation for mass transfer of component A, without chemical reaction, in the z-direction

$$\frac{\partial \bar{c}_A}{\partial t} + U\frac{\partial \bar{c}_A}{\partial z} = \frac{\partial}{\partial z}\left(E_L \frac{\partial \bar{c}_A}{\partial z}\right) \tag{1}$$

where $\bar{c}_A$ is the time-averaged concentration of A and U the mean velocity (Daily and Harleman, 1966, for single-phase fluid). Often, when Eq. 1 is used to discuss experimental data, E_L, is assumed to be constant.

The value of E_L depends on fluid motion in the bubble column, which in turn depends on flow rates, column dimensions, configuration of fluid injection and properties of fluids. Published predictions of E_L either have not taken into account physical properties and fluid injection arrangements, or have included the fractional gas content (holdup) e, which is a quantity that may depend on all of the above-listed variables. However, the relation between e and E_L is uncertain (e.g., Towell and Ackerman, 1972), there is disagreement between correlations for e (e.g., Hikita et al., 1980) and doubt about their applicability to columns of diameters larger than those providing the correlation data, even measured values of gas holdup are suspect (Table 5, Field and Davidson, 1980). Improved accuracy in prediction of E_L is also desirable.

PREDICTION OF E_L FROM FLOWRATE AND EQUIPMENT CONFIGURATION

In bubble columns, axial dispersion is the result of eddies and of velocity profiles that are created by the relatively fast upward motion of the large bubbles. In simple bubble columns, bubble drag force is the major indicator of eddy creation. Turbulent mixing, i.e., mixing by eddy motion, has been correlated using the quantity (force/density) (e.g., Fossett and Prosser, 1949; Taylor, 1954; Fox and Gex, 1956). For single bubbles, the drag force is a function of bubble size, properties of fluids and bubble velocity. The velocity of single bubbles of size D rising through liquid in tubes of diameter d decreases when the perimeter ratio D/d increases.

In a given system, there is a value of D/d above which bubbles have cylindrical shape parallel to the tube wall and their terminal velocity depends on d only (Maneri and Mendelson, 1968; Wallis, 1969-cited by Clift et al., 1978). Maneri and Mendelson (1968) proposed that for systems in which $(\rho_c g d^2/4\sigma)$ is large,

$$\frac{\text{rise velocity of bubble in tube of dia. } d}{\text{rise velocity of bubble in infinite liquid}} = \left(\tanh \frac{0.25d}{D}\right)^{1/2}, \quad \frac{\rho_c g d^2}{4\sigma} \to \infty \tag{2}$$

Equation 2 suggests a way of relating the magnitude of E_L to that of column diameter d using a hindered motion factor B which contains an appropriate perimeter ratio.

Axial dispersion will be enhanced also by the axial component of fluid injection, the resulting contribution to E_L being E_K say and by velocity in excess of v_{sg}. The proposed equation for the turbulent axial dispersion coefficient is then

$$E_L = E_B B + \sum E_K - (Q v_{sg})^{1/2} \tag{3}$$

where E_B = (bubble drag force in infinite liquid/density of liquid)$^{1/2}$
Q = volumetric flowrate of gas
v_{sg} = superficial gas velocity = $4Q/(\pi d^2)$

Because axial dispersion is related to the upward motion of large bubbles, the behaviour of the small coexisting, often recirculating bubbles which enter discussion of holdup and interfacial area is not considered here. In Eq. 3, Q and v_{sg} are found from basic data, but methods for evaluation of E_B, B and E_K are to be devised as follows.

At the gas flow rates usually applied in bubble columns, bubbles form continuously at each sparger orifice and bubble size increases with flow rate. When the flow rate has increased to the jet regime, fine bubbles formed near the orifice decelerate and periodically form a coalesced bubble within a short distance. Thus, the existence of notional bubble chains can be envisaged.

Fluid motion in bubble columns has been described (e.g., Towell et al. 1965; Argo and Cova 1965). From these, it appears that unless the bubbles move without vortex formation, sparger orifices are widely spaced, or the system contains material that hinders coalescence, chain bubbles will collide to form larger aggregates, these will fragment and recoalesce.

With gas flow through N orifices, readily calculable limits of size of the large rising bubbles are:

D_{BO} = size of chain bubble formed at each of N orifices
D_{BMAX} = size of chain bubble when the total gas flow is carried by a single chain

Then, from a steady-state force balance and Eq. 3, for the chain bubbles

$$E_B = N(C_{DBO}\pi D_{BO}^2 v_{BO}^2/8)^{1/2} \tag{4}$$

where C_{DBO} is the drag coefficient and v_{BO} is the terminal velocity in free motion. In view of repeated coalescence and breakup, the geometric mean of ND_{BO} and D_{BMAX} is a suitable measure in a turbulent bubble bed. Substituting in Eq. 4,

$$E_B = ((\pi/8)NC_{DBO}^{1/2}D_{BO}v_{BO}C_{DBMAX}^{1/2}D_{BMAX}v_{BMAX})^{1/2} \tag{5}$$

Evaluation of the hindered motion factor B in Eq. 3 is based on Maneri and Mendelson's (1968) analogy to wave velocity in shallow water, i.e., Eq. 2. It is assumed here that a similar retarding pe-

TABLE 1. COMPARISON OF OBSERVED AND CALCULATED LIQUID-PHASE AXIAL DISPERSION COEFFICIENTS E_L IN AIR-WATER BUBBLE COLUMNS WITHOUT INTERNAL FITTINGS

Observed (Towell and Ackerman 1972)					Calculated E_L m^2s^{-1}			$[E_L$ (Calculated) − E_L (Observed)] 100 / E_L (Calculated)		
Converted Rounded-Off Values Only				Converted from lb, ft Units:						
d m	z_{total}^* m	v_{sg} ms^{-1}	v_{sc} ms^{-1}	E_L m^2s^{-1}	Eq. 15 (Field and Davidson)	Eq. 16 (Deckwer et al.)	Eq. 14 (This Work)	Eq. 15 Field and Davidson)	Eq. 16 (Deckwer et al.)	Eq. 14 (This Work)
From Table 1, Liquid Back-Mixing Data										
0.406	1.500	0.0219	0.0068	0.0671	0.0529	0.0611	0.0651	−27	−10	−3
0.406	1.500	0.0884	0.0068	0.0929	0.0957	0.0928	0.0977	+3	0	+5
0.406	1.500	0.0219	0.0136	0.0361	0.0389	0.0611	0.0651	+7	+41	+45
0.406	1.500	0.0884	0.0136	0.0826	0.0847	0.0928	0.0977	+2	+11	+15
0.406	2.845	0.0192	0.0068	0.0800	0.0569	0.0587	0.0621	−41	−36	−29
0.406	2.845	0.0762	0.0068	0.0877	0.1069	0.0888	0.0954	+18	+1	+8
0.406	2.845	0.0192	0.0136	0.0542	0.0463	0.0587	0.0621	−17	+8	+13
0.406	2.845	0.0762	0.0136	0.0955	0.0989	0.0888	0.0954	+3	−8	0
1.067	5.105	0.0174	0.0037	0.2890	0.2535	0.2200	0.2434	−14	−31	−19
1.067	5.105	0.0347	0.0037	0.2322	0.3194	0.2709	0.3130	+27	+14	+26
1.067	5.105	0.0085	0.0072	0.1884	0.1948	0.1778	0.1846	+3	−6	−2
1.067	5.105	0.0174	0.0072	0.3252	0.2535	0.2200	0.2434	−28	−48	−34
1.067	5.105	0.0347		0.2477	0.2754	0.2709	0.3130	+10	+9	+21
1.067	5.105	0.0174		0.2245	0.2535	0.2200	0.2434	+11	−2	+8
								Avg. = 15	Avg. = 16	Avg. = 16
From Table 2, Liquid Pulse Data										
0.406	1.803	0.2682	0.0122	0.0774	0.1665	0.1295	0.0760	+54	+40	−2
0.406	1.803	0.0671	0.0122	0.0645	0.0960	0.0854	0.0703	+33	+24	+8
0.406	1.803	0.0168	0.0122	0.0361	0.0605	0.0564	0.0422	+40	+36	+14
0.406	1.803	0.0168	0.0073	0.0297	0.0605	0.0564	0.0422	+51	+47	+30
0.406	1.803	0.0671	0.0073	0.0645	0.0960	0.0854	0.0703	+33	+24	+8
0.406	1.803	0	0.0025	0.0015	0	0				
0.406	1.500	0.0204	0.0068	0.0413	0.0550	0.0598	0.0635	+25	+31	+35
0.406	1.500	0.0204	0.0136	0.0284	0.0310	0.0598	0.0635	+8	+53	+55
0.406	2.845	0.0189	0.0136	0.0465	0.0153	0.0584	0.0617	−204	+20	+25
0.406	2.845	0.0189	0.0068	0.0800	0.0558	0.0584	0.0617	−43	−37	−30
								Avg. = 55	Avg. = 35	Avg. = 23

* z_c is calculated from z_{total} and experimental holdup data given in the paper for each run.

rimeter effect exists in bubble columns. In addition, it can be expected that axial liquid circulation decreases when ungassed liquid height z_c decreases. A complete factor for hindered motion is then

$$B = \left(\tanh \frac{0.25d}{(ND_{BO}D_{BMAX})^{1/2}}\right)^{1/2} f(z_c/D_{BMAX}),$$

$$f(z_c/D_{BMAX}) = 1 \text{ when } z_c \gg D_{BMAX} \quad (6)$$

To find the approximate magnitude of ΣE_K in Eq. 4, it is assumed that only axial components of momentum contribute to axial dispersion.

$$\sum E_K = \sum(\rho/\rho_c)^{1/2}(Qv)^{1/2}_{0,\text{AXIAL}} \quad (7)$$

The effect of liquid throughflow on E_L, often said to negligible, is difficult to quantify. When there is no gas flow, axial dispersion is unlikely to conform to models based on developed velocity profiles. Countercurrent mean downward flow of liquid increased gas holdup in 0.05 m diameter tubes compared with cocurrent flow (Otake et al., 1981). Generally, mixing would be greater than with cocurrent flow. Quantitative prediction of the effect of magnitude and direction of superficial liquid velocity on E_L is not attempted here and may be unimportant at the usual operating conditions of simple bubble columns.

Further uncertainties exist in comparing predicted with observed E_L when description of equipment is incomplete; this applies particularly to configuration of fluids injection. Without such data, D_{BO} and $(Qv)^{1/2}_{0,\text{AXIAL}}$ cannot be calculated.

Predictions of E_L by Eq. 3 are shown for comparison with experimental values and with other predictions in Tables 1 and 2. In the systems considered there, the chain bubble concept can be used. In free motion, a determinate velocity of the chain bubble is v_B, which is the terminal velocity of a single bubble. Then

$$Q = N(\pi/6)D_{BO}^3(\text{bubbles/time}) = N(\pi/6)D_{BO}^2 v_B \quad (8)$$

$$v_B = \left(\frac{4(\rho_c - \rho_d)gD}{3\rho_c C_D}\right)^{1/2},$$

$$\rightarrow (gD/2)^{1/2} \text{ when } C_D \rightarrow 8/3, \quad \rho_c \gg \rho_d \quad (9)$$

From Eqs. 8 and 9, with $\rho_c \gg \rho_d$ as is usual in bubble columns,

$$D_{BO} = (6/\pi)^{2/5}(Q/N)^{2/5}(3C_D/4g)^{1/5} \quad (10)$$

Considering Q and N in the systems discussed, the shapes of the bubble drag coefficient curves and a transition criterion to viscosity-independent bubble motion (Lehrer, 1980), with C_D in the viscosity-independent regime given by

$$C_D = 8E\ddot{o}/3(6 + E\ddot{o}), \quad \rightarrow 8/3 \text{ when } E\ddot{o} \rightarrow \infty \quad (11)$$

it is a valid approximation to use here, from Eqs. 9, 10 and 11,

$$C_D = 8/3, \quad v_B = (gD/2)^{1/2},$$

$$D_{BO} = (6/\pi)^{2/5}(2/g)^{1/5}(Q/N)^{2/5} \quad (12)$$

Values of D_{BO} calculated by Eq. 12 are 7.8% larger than those calculated by the model of Davidson and Schüler (1960) for bubble formation in inviscid liquid. With regard to the work of Towell and Ackerman (1972), only the sparger for $N = 1$ is detailed. For the two-fluid spargers, $N = 19$ and $N = 3(41)$, without details of arrangement. The atomising action of two-fluid spargers and their possible configuration (Towell et al., 1965) would tend to reduce ΣE_K in Eq. 3, possibly making it insignificant. The downward injection from a single gas orifice near the vessel floor and an impingement plate opposite the liquid inlet result in $\Sigma E_K = 0$. D_{BMAX}

TABLE 2. COMPARISON OF OBSERVED AND CALCULATED VALUES OF $_L$ IN COLUMNS WITH UPWARD FACING SPARGERS

N	d_O mm	v_{sg} m·s^{-1}	E_L, m^2·s^{-1} observed	Eq. 15*	Eq. 16	This work, Eq. 17
			Ohki and Inoue (1970), d = 0.16 m			
37	0.4	0.05	0.0390	0.0179	0.0212	0.0417
37	2	0.05	0.0270	0.0232	0.0212	0.0283
37	2	0.15	0.0360	0.0389	0.0295	0.0388
37	1	0.15	0.0350	0.0389	0.0295	0.0489
37	2	0.24	0.0410	0.0464	0.0334	0.0425
91	0.4	0.05	0.0480	imaginary	0.0212	0.0373
91	2	0.05	0.0275	0.0232	0.0212	0.0287
91	1	0.15	0.0340	0.0389	0.0295	0.0440
91	2	0.15	0.0350	0.0389	0.0295	0.0375
91	2	0.24	0.0430	0.0464	0.0334	0.0396
			Deckwer et al. (1974), d = 0.20 m			
56	1	0.03	0.0295	0.0331	0.0249	0.0337
56	1	0.12	0.0430	0.0682	0.0377	0.0616

* Values of e from data of Ohki and Inoue (1970) and Deckwer et al. (1974).

is calculated from Eq. 12 with $N = 1$. With $z_c > 5D_{BMAX}$, $f(z_c/D_{BMAX})$ in Eq. 6 can be taken as unity. Therefore, for Table 1,

$$\Sigma E_K = 0, \quad f(z_c/D_{BMAX}) = 1 \tag{13}$$

From Eqs. 3, 6, 12 and 13,

$$E_L = 1.3134 g^{0.2} N^{0.2} Q^{0.6} \times \left(\tanh \frac{0.168 g^{0.2} d}{N^{0.3} Q^{0.4}}\right)^{1/2} - (Q v_{sg})^{1/2} \tag{14}$$

Field and Davidson (1980) have used data from Towell and Ackerman's (1972) work in a comparison of predictive equations. Among these, frequently cited are:

$$E_L = 0.90 d^{1.5} (z_{\text{total}}(v_{sg} - 0.235e))^{1/3} \text{ m}^2\text{·s}^{-1} \tag{15}$$

$$E_L = 0.678 d^{1.4} v_{sg}^{0.3} m^2 \cdot s^{-1} \tag{16}$$

In Table 1, Eq. 14 provides better agreement between prediction and experiment than Eqs. 15 and 16; Eq. 14 also provides better agreement than Riquart's (1981) equation for E_L. The experimental data of Ohki and Inoue (1970) pertain to columns with upward-facing gas orifices, and the same can be assumed for those of Deckwer et al. (1974). Using these data, Table 2 shows observed and predicted values of E_L. The comparison uses Eq. 3 with $E_B B$ as in Eq. 14 and E_K, Eq. 7, in its simplest form, so that

$$E_L = 1.3134 g^{0.2} N^{0.2} Q^{0.6} \left(\tanh \frac{0.168 g^{0.2} d}{N^{0.3} Q^{0.4}}\right)^{1/2} + \left(\frac{\rho_d}{\rho_c}\right)^{1/2} \left(\frac{4Q^2}{N\pi d_0^2}\right)^{1/2} - (Q v_{sg})^{1/2} \tag{17}$$

Equation 17 compares favorably with Eqs. 15 and 16.

CONCLUSION

A simple model has been formulated to evaluate the turbulent liquid axial dispersion coefficient E_L in a simple two-phase bubble column. For comparison with previously-reported work, the simplified Eqs. 14 and 17 could be used; they allow direct calculation without recourse to further correlations and factors. Considering the vagaries of turbulent fluid motion in bubble columns, predictive ability is satisfactory. Physical properties can be taken into account via C_D, starting with Eq. 9. In noncoalescing systems, bubbles which fragment may not recoalesce, so that a $D_{BMINIMUM}$ instead of D_{BMAX} could be appropriate; possibly $D_{BMINIMUM}$ is $(6\sigma/\Delta\rho g)^{1/2}$, experimental values are required for comparison. The upper limit of validity of the model is when $E_L = 0$ at large Q in Eq. 14; then

$$v_{sg} \leq 0.37 N^{1/12} (gd)^{1/2},$$
$$\text{e.g., when } N = 1, \quad D_{BMAX}/d = 0.908 \tag{18}$$

The lower limit in systems containing gas in inviscid liquid is near $v_{sg} = 10$ mm·s^{-1}, a rational quantitative criterion is apparently still lacking. Equal number of sparger orifices at wider pitch result in greater homogeneity of the bubble bed at equal superficial velocity; extension of the model to allow for an effect on E_L requires further experimental data.

NOTATION

B = hindered motion factor, defined by Eq. 6
C_D = drag coefficient of single bubble in free motion
d = inside diameter of column, L
d_O = diameter of sparger orifice, L
D = bubble size $((6/\pi)\text{volume})^{1/3}$, L
e = fractional gas content of two-phase fluid
E_B = component of E_L, defined by Eqs. 4 and 5, L^{2}t^{-1}
E_K = component of E_L, defined by Eq. 7, L^{2}t^{-1}
E_L = axial turbulent dispersion coefficient in continuous liquid phase, L^{2}t^{-1}
$E\ddot{o}$ = Eötvös number = $\Delta\rho g D^2/\sigma$
g = gravity acceleration, Lt^{-2}
N = number of sparger holes
Q = mean volumetric gas flow rate, L^{3}t^{-1}
v_B = terminal velocity of single bubble in free motion, Lt^{-1}
v_{sg} = superficial velocity of gas in column = $4Q/(\pi d^2)$, Lt^{-1}
z_{total} = height of two-phase fluid, L
z_c = height of liquid at rest, L

Greek Letters

ρ = density of fluid, ML^{-3}
σ = interfacial tension, Mt^{-2}

Subscripts

c = continuous phase
d = dispersed phase
BO = bubble formed at sparger orifice
$BMAX$ = largest bubble
O = at injection orifice

LITERATURE CITED

Argo, W. B., and D. R. Cova, "Longitudinal Mixing in Gas-sparged Tubular Vessels," *I & EC Process Design Dev.*, 4, 4, p. 352 (1965).
Baird, M. H. I., and R. G. Rice, "Correspondence," *Trans. I. Chem. E.*, **59** (1981).

Clift, R., J. R. Grace, and M. E. Weber, *Bubbles, Drops and Particles*, Academic Press (1978).

Daily, J. W., and D. R. F. Harleman, *Fluid Mechanics*, Addison-Wesley (1966).

Davidson, J. F., and Schüler, "Bubble Formation at an Orifice in an Inviscid Liquid," *Trans. I. Chem. E.*, **38**, p. 335 (1960).

Deckwer, W.-D., R. Burckhart, and G. Zoll, "Mixing and Mass Transfer in Tall Bubble Columns," *Chem. Eng. Sci.*, **29**, p. 2177 (1974).

Field, R. W., and J. F. Davidson, "Axial Dispersion in Bubble Columns," *Trans. I. Chem. E.*, **58**, p. 228 (1980).

Fossett, H., and L. E. Prosser, "The Application of Free Jets to the Mixing of Fluids in Bulk," *Proc. I. Mech. E.*, **160**, p. 224 (1949).

Fox, E. A., and V. E. Gex, *A.I. Chem. E.J.*, **2**, p. 539 (1956).

Hikita, H., S. Asai, K. Tanigawa, K. Segava, and M. Kitao, "Gas Hold-up in Bubble Columns," *The Chem. Eng. J.*, **20**, p. 59 (1980).

Joshi, J. B., "Axial Mixing in Multiple Contactors—A Unified Correlation," *Trans. I. Chem. E.*, **58**, p. 155 (1980).

Joshi, J. B., "Correspondence," *Trans. I. Chem. E.*, **59**, p. 139 (1981).

Joshi, J. B., and Y. T. Shah, "Hydrodynamic and Mixing Models for Bubble Column Reactors," *Chem. Eng. Commun.*, **11**, p. 165 (1981).

Lehrer, I. H., "A Theoretical Criterion in the Free Motion of Single Bubbles and Drops," *A. I. Chem. E. J.*, **26**, p. 170 (1980).

Maneri, C. C., and H. D. Mendelson, "The Rise Velocity of Bubbles in Tubes and Rectangular Channels as Predicted by Wave Theory," *A. I. Chem. E. J.*, **14**, p. 295 (1968).

Mersmann, A., "Auslegung und Maszstabvergrösserung von Blasen-und Tropfensäulen," *Chem. Ing. Tech.*, **49**, p. 679 (1977).

Ohki, Y., and H. Inoue, "Longitudinal mixing of the liquid phase in bubble columns," *Chem. Eng. Sci.*, **25**, p. 1 (1970).

Otake, T., S. Tone, and K. Shinohara, "Gas Holdup in the Bubble Column with Cocurrent and Countercurrent Gas-Liquid Flow," *J. Chem. Eng. Japan*, **13**, p. 338 (1981).

Riquarts, H. P., "Strömungsprofile, Impulsaustausch und Durchmischung der flüssigen Phase in Blasensäulen," *Chem. Ing. Tech.*, **53**, p. 60 (1981).

Shah, Y. T., G. J. Stiegel, and M. M. Sharma, "Backmixing in Gas-Liquid Reactor," *A. I. Chem. E. J.*, **24**, p. 369 (1978).

Taylor, G., "The Dispersion of Matter in Turbulent Flow Through a Pipe," *Proc.*, Royal Soc. London, Ser. A, **223**, p. 446 (1954).

Towell, G. D., C. P. Strand, and G. H. Ackerman, "Mixing and Mass Transfer in Large Diameter Bubble Columns," *Proc. Symp.*, Mixing—Theory Related To Practice, AIChE-I.Chem.E. Symp. Ser. No. 10. p. 97 (1965).

Towell, G. D., and G. H. Ackerman, "Axial Mixing of Liquid and Gas in Large Bubble Reactors," *Proc.*, Second Int'l. Sympos. Chem. Reaction Eng., Elsevier, Amsterdam (1972).

Wallis, G. B., *One-Dimensional Two-Phase Flow*, McGraw-Hill (1969).

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Curve-Fitting Monotonic Functions

K. VISWANATHAN

Department of Chemical Engineering
Indian Institute of Technology,
Hauz Khas, New Delhi 110016, India

INTRODUCTION

The identification of the form of the functional relationship between variables is generally approached by trial and error. Some standard forms of functions are tried for a particular shape, and the one that deviates least from the actual values, experimentally obtained, is chosen (Davies, 1962). Or, a least square polynomial is fitted. Once the form of the function is known, the problem at hand can be solved analytically.

The method given in this paper can be used to avoid excessive trial and error in arriving at a functional form. The method is developed for functions known to be monotonic which is often the case in engineering examples. In such cases polynomials are not suitable as they may not everywhere have a positive (or negative) slope. The analysis described here can be applied when theoretical expressions are not available. In the present study, only one independant variable and one dependant variable are considered.

THEORETICAL DEVELOPMENT

A new variable Z is defined as

$$Z = \frac{x}{y}\frac{dy}{dx} = \frac{xy'}{y} \tag{1}$$

Z is the ratio of fractional change in y to that of x. It will be called the monotonic transform of y with respect to x and will be written as $\mathrm{mon}_x\, y$ (to be read as mon y to the base x) or simply mon y. Some standard forms of monotonically increasing functions, Table 1, are considered. Table 1 also gives the monotonic transforms of the various functions.

Our aim is to calculate Z from experimentally-obtained x and y values. For this, the following finite difference approximation which follows from Eq. 1 is used.

$$Z\Bigg|_{x = \frac{x_i + x_{i+1}}{2}} = \frac{ln(y_{i+1}/y_i)}{ln(x_{i+1}/x_i)} \tag{2}$$

It can be shown that Eq. 2 would be exact for

$$ln\, y = a(ln\, x)^2 + m\, ln\, x + ln\, C$$

or

$$y = Cx^{m + a l\, ln\, x}$$

TABLE 1. SOME FREQUENTLY OCCURING MONOTONICALLY INCREASING FUNCTIONS AND THEIR MONOTONIC TRANSFORMS

Name	Function	Monotonic Transform
Basic 1	$y = Cx^m$	$Z = m$
Basic 2	$y^* = C\exp(bx^n)$	$Z = nbx^n$
Basic 3	$y = C\exp(bx^{-n})$	$Z = nbx^{-n}$
Basic 4	$y = \left(\frac{1}{b}\, ln\, \frac{x}{C}\right)^{1/n}$	$Z = \frac{1}{n ln(x/C)}$
Basic 5	$y = \left(-\frac{1}{b}\, ln\, \frac{x}{C}\right)^{-1/n}$	$Z = \frac{-1}{n ln(x/C)}$
Nonbasic 1	$y = Cx^m \exp(bx^n)$	$Z = m + nbx^n$
Nonbasic 2	$y = Cx^m \exp(-bx^{-n})$	$Z = m + nbx^{-n}$
Nonbasic 3	$y = Cx^m(lnbx)^{1/n}$	$Z = m + \frac{C^n x^{mn}}{ny^n}$
Nonbasic 4	$y = Cx^m(-lnbx)^{-1/n}$	$Z = m + \frac{C^{-n} x^{-mn}}{ny^{-n}}$

* The general form $CK^{b'x^n}$ is included here where b equals $b'lnK$.